

Exercises 2.1 page 43

(13) A couple of broke students.....

Solution

(i) Given demand function

$$p = 50 - 0.05q$$

we know that $R = pq$

$$\Rightarrow R = (50 - 0.05q)(q)$$

$$\Rightarrow R = -0.05q^2 + 50q ; 50 < q < 370$$

Given $F = \$100$ and variable cost, $v = 1 + 0.02q$

we know that the total cost

$$C = F + vq$$

$$\Rightarrow C = 100 + (1 + 0.02q)(q)$$

$$\Rightarrow C = 0.02q^2 + q + 100 ; 50 < q < 370$$

(ii) From $R(q) = -0.05q^2 + 50q$

$$\Rightarrow MR(q) = \lim_{h \rightarrow 0} \frac{R(q+h) - R(q)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(-0.05(q+h)^2 + 50(q+h)) - (-0.05q^2 + 50q)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-0.05(q^2 + 2qh + h^2) + 50q + 50h + 0.05q^2 - 50q}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-0.05q^2 - 0.1qh - 0.05h^2 + 50h + 0.05q^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-0.1qh - 0.05h^2 + 50h}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-0.1q - 0.05h + 50}{1}$$

$$= \lim_{h \rightarrow 0} (-0.1q - 0.05h + 50)$$

$$= -0.1q - 0.05(0) + 50$$

$$\Rightarrow MR(q) = -0.1q + 50$$

From $C(q) = 0.02q^2 + q + 100$

$$\Rightarrow MC(q) = \lim_{h \rightarrow 0} \frac{C(q+h) - C(q)}{h}$$

$$\Rightarrow MC(q) = \lim_{h \rightarrow 0} \frac{(0.02(q+h)^2 + (q+h) + 100) - (0.02q^2 + q + 100)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{0.02(q^2 + 2qh + h^2) + q + h + 100 - 0.02q^2 - q - 100}{h}$$

$$= \lim_{h \rightarrow 0} \frac{0.02q^2 + 0.04qh + 0.02h^2 + h - 0.02q^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{0.04qh + 0.02h^2 + h}{h}$$

$$= \lim_{h \rightarrow 0} (0.04q + 0.02h + 1)$$

$$= 0.04q + 0.02(0) + 1$$

$$\Rightarrow MC(q) = 0.04q + 1$$

(iii) When $q = 300$, $MR(q) = 50 - 0.1q$
 $MR(300) = 50 - 0.1(300) \checkmark$
 $= \$20$
 and $MC(300) = 1 + 0.04(300) \checkmark MC(q) = 1 + 0.04q$
 $= \$13$
 At $q = 300$, we have
 $MC(300) < MR(300)$

$\therefore$ We should increase production to increase profit since $MC < MR$.

(iv) Profit is maximised when

$$MC(q) = MR(q)$$

$$\Rightarrow 0.04q + 1 = -0.1q + 50$$

$$\Rightarrow 0.04q + 0.1q = 50 - 1$$

$$\Rightarrow 0.14q = 49$$

$$\Rightarrow q = \frac{49}{0.14} = 350$$

This is acceptable since it is on the domain.

$\therefore$ Profit is maximised when output is $q = 350$ cupcakes.

2.2 The Limit of a Function

We saw in the previous section, that we can perform a direct substitution to a polynomial

Example

$$\lim_{h \rightarrow 0} (a_0 + a_1 h + a_2 h^2)$$

$$= a_0 + a_1 \cdot 0 + a_2 \cdot 0$$

$$= a_0 + a_1$$

Note:- $h \rightarrow 0$ is not the same as $h = 0$.

$h \rightarrow 0$ means h gets very close to 0 but does not get to become 0.

This is a mathematical case of "so close yet so far".

Therefore, when we write

$$\lim_{x \rightarrow a} f(x) = L \text{ means } x \text{ approaches } a \text{ but does not become } a$$

Properties

(1) $\lim_{x \rightarrow a} k = k$; k is a constant

(2) $\lim_{x \rightarrow a} x = a$; Direct substitution

(3) $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$

(4) $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$

(5) $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$; provided $\lim_{x \rightarrow a} g(x) \neq 0$

$$(6) \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$$

Example Ex 2.2

$$(13) \lim_{x \rightarrow -6} \sqrt{x^2 + 1} = \sqrt{\lim_{x \rightarrow -6} (x^2 + 1)}$$

$$= \sqrt{(-6)^2 + 1} = \sqrt{37}$$

Basically a Direct Substitution!

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$$(10) \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \frac{2-2}{2^2-4} = \frac{0}{0}$$

Very Important

$\frac{0}{0}$ is indeterminate

$\pm \frac{\infty}{\infty}$ is also indeterminate

In some cases, direct substitution does not work!

So we need to tweak the expression.

$$\lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{\cancel{x-2}}{(\cancel{x-2})(x+2)}$$

$$= \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{2+2} = \frac{1}{4}$$

$$(20) \lim_{x \rightarrow 11} \frac{11-x}{-x+11} = \lim_{x \rightarrow 11} \frac{11-x}{11-x}$$

$$= \lim_{x \rightarrow 11} 1 = 1$$

$$(15) \lim_{x \rightarrow 0} \frac{(x+1)^2 - 1}{x} = \lim_{x \rightarrow 0} \frac{x^2 + 2x + 1 - 1}{x}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 + 2x}{x} = \lim_{x \rightarrow 0} \frac{x(x+2)}{x}$$

$$= \lim_{x \rightarrow 0} (x+2) = 0+2 = 2$$

$$(26) \lim_{x \rightarrow 25} \frac{5-\sqrt{x}}{25-x} = \lim_{x \rightarrow 25} \frac{5-\sqrt{x}}{5^2-(\sqrt{x})^2} \quad \text{since } x > 0$$

$$= \lim_{x \rightarrow 25} \frac{5-\sqrt{x}}{(5-\sqrt{x})(5+\sqrt{x})}$$

$$= \lim_{x \rightarrow 25} \frac{1}{5+\sqrt{x}}$$

$$= \frac{1}{5+\sqrt{25}} = \frac{1}{5+5} = \frac{1}{10}$$

OR we multiply and divide by the conjugate of $5-\sqrt{x}$.

What is a conjugate? Take one of the terms and change its sign.

① Conjugate of $a+b$ is $a-b$ or $-a+b$

② Conjugate of $x-y$ is $x+y$ or $-x-y$

③ Conjugate of $0+a$ is $-0+a = 0+a$

Example:- Conjugate of $\sqrt{x}$ is also $\sqrt{x}$ or $-\sqrt{x}$

$$\begin{aligned}
 \textcircled{26} \quad \lim_{x \rightarrow 25} \frac{5 - \sqrt{x}}{25 - x} &= \lim_{x \rightarrow 25} \frac{(5 - \sqrt{x})(5 + \sqrt{x})}{(25 - x)(5 + \sqrt{x})} \\
 &= \lim_{x \rightarrow 25} \frac{5^2 - (\sqrt{x})^2}{(25 - x)(5 + \sqrt{x})} \\
 &= \lim_{x \rightarrow 25} \frac{25 - x}{(25 - x)(5 + \sqrt{x})} \quad \leftarrow \begin{array}{l} \text{since } x \rightarrow 25 \\ \text{we know } x > 0 \\ \text{and therefore} \\ (\sqrt{x})^2 = x \end{array} \\
 &= \lim_{x \rightarrow 25} \frac{1}{5 + \sqrt{x}} \\
 &= \frac{1}{5 + \sqrt{25}} = \frac{1}{5 + 5} = \frac{1}{10}
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{27} \quad \lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{2 - x} &= \lim_{x \rightarrow 2} \frac{\frac{2}{2x} - \frac{x}{2x}}{2 - x} \\
 &= \lim_{x \rightarrow 2} \frac{\frac{2-x}{2x}}{2-x} \\
 &= \lim_{x \rightarrow 2} \frac{2-x}{2x(2-x)} \\
 &= \lim_{x \rightarrow 2} \frac{1}{2x} = \frac{1}{(2)(2)} = \frac{1}{4}
 \end{aligned}$$

Note

$$\begin{aligned}
 \frac{\frac{a}{b}}{\frac{c}{d}} &= \frac{a}{b} \div \frac{c}{d} \\
 &= \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}
 \end{aligned}$$

$$\begin{aligned}
 \text{So } \lim_{x \rightarrow 2} \frac{\frac{2-x}{2x}}{2-x} &= \lim_{x \rightarrow 2} \frac{\frac{2-x}{2x}}{\frac{2-x}{1}} \\
 &= \lim_{x \rightarrow 2} \frac{(2-x)(1)}{(2x)(2-x)}
 \end{aligned}$$